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CROSSING-SYMMETRIC RINGS, LADDERS, AND EXCHANGES

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A new approach is used in the description of fermion and boson systems at zero or finite temperature. We generalize the familiar ladder and chaining operations to construct a crossing-symmetric approximation to the two-particle vertex from the bare interaction. The explicit rules for this construction are given in terms of Feynman propagators. The high level of symmetry of the vertex simplifies the theory. In one form, the structure appears as a generalization of parquet diagrams. The energy, self-energy, and vertex satisfy a number of consistency relations.

Previous work has described in detail the intimate relationship between optimized Jastrow variational theory and perturbation theory [1]. Although that paper focused on boson systems, it explicitly indicated the role that crossing symmetry must play in a minimally consistent description of fermion systems and systems at finite temperature. Guided by the work of refs. [2,3], we develop here a crossing-symmetric perturbation theory for the two-particle vertex, the single-particle self-energy, and the energy.

Green's functions made by attaching legs to the possible forms of the two-particle vertex, Γ, are shown in fig. 1a. *Reducible* vertices of type S, T, and U may be divided into two disconnected parts by cutting two propagator lines as shown in fig. 1d. The type I diagrams are not reducible in any channel; the bare interaction V, is of this type. In terms of the channel operations, s, t, and u, the crossing-symmetric equations take the form [1]

$$\Gamma = \mathrm{I} + \mathrm{S} + \mathrm{T} + \mathrm{U}\,, \tag{1}$$

$$\mathrm{S} = (\mathrm{I} + \mathrm{T} + \mathrm{U})\, G_{\mathrm{s}}\, \Gamma\,, \tag{2}$$

$$\mathrm{T} = (\mathrm{I} + \mathrm{S} + \mathrm{U})\, G_{\mathrm{t}}\, \Gamma\,, \tag{3}$$

$$\mathrm{U} = (\mathrm{I} + \mathrm{S} + \mathrm{T})\, G_{\mathrm{u}}\, \Gamma\,, \tag{4}$$

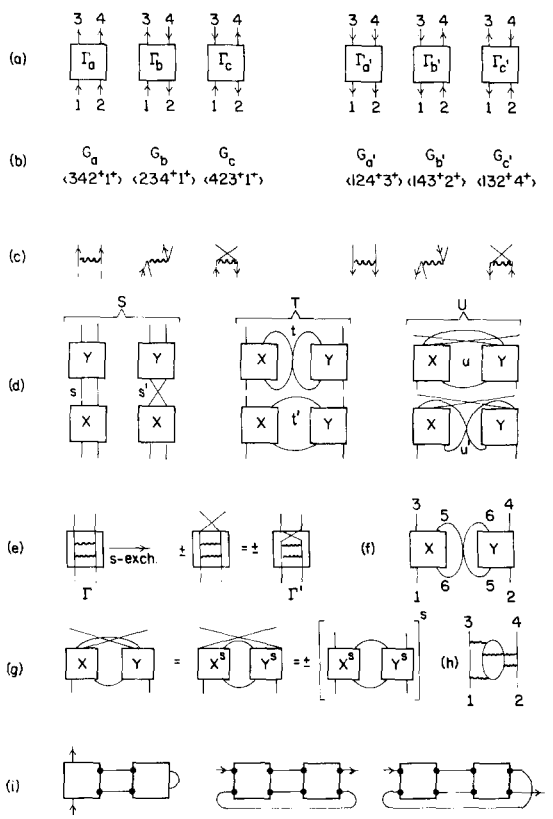

Fig. 1. Illustrations of points made in the text.

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where G_x represents a pair of propagators in the appropriate channel. Equivalent crossing symmetric equations were constructed by refs. [2,3] as well as ref. [4]. The present form exposes the crossing symmetry quite clearly; in addition, approximation schemes similar to those in ref. [1] will directly involve the channel operations used in constructing the reducible diagrams S, T, and U.

We begin with some conventions. The ± which appears frequently takes on the value + for bosons, − for fermions. The structure of perturbation theory is identical at $T = 0$ and at finite temperatures, but since the Green's function is defined differently in these two theories [5] and keeping variable factors of −1 or i throughout would prove tedious, we will use the one-body Green's function

$$G_{\alpha\beta}(x, x') = p\langle \mathrm{T}\{\psi_\alpha(x)\psi^+_\beta(x')\}\rangle . \tag{5}$$

The factor $p = -\mathrm{i}$ ($T = 0$), -1 ($T \neq 0$), or 1 (our case), the angular brackets denote an appropriate expectation value, and the T{ } denotes a suitable ordering operator. The choice of p does not affect the topological structure of loops (including the factors of −1 for each fermion loop); the different factors of i ($T = 0$), (−1) ($T \neq 0$) or 1 (our choice) can easily be introduced when needed. Spin indices will be suppressed.

The six 2-body Green's functions, $G_x(1, 2, 3, 4)$, are drawn in fig. 1a and defined in fig. 1b with implicit ordering. Each is identified with a $\Gamma_x(1, 2, 3, 4)$; the (anti)symmetry of each Green's function under interchange of outgoing lines is to be associated with the Γ. The *direct* irreducible vertex contributions associated with the bare interaction are shown in fig. 1c. The *exchange* contributions to each Γ depend on which pair of lines are being exchanged (using the crossing symmetries we can exchange ingoing and outgoing lines). From the definitions of the various Green's functions, we compute the effect on a given diagram type of s-exchange (3 ↔ 4), t-exchange (2 ↔ 4), and u-exchange (2 ↔ 3). Graphically, the exchange of a given diagram is performed by rearranging the legs of the Green's function, as illustrated in fig. 1e.

The corresponding Γ′ can be found by redrawing the inside of the box and letting the legs come out in the canonical order. The exchange operations convert vertex diagrams from one type (a, b, c) to another as noted in table 1. The exchanges of the I diagrams in the x-channel will be denoted by I^x.

We return now to the channel operations (s, s′, t, t′, u, u′) of fig. 1d which connect two Γ's together with propagators to form a new Γ. The direction of each propagator joining two Γ's will depend on the types (a, b, c, a′, b′, c′) of the Γ's involved and on the operation since it must connect an outgoing point to an ingoing point. An overall ± factor may arise; e.g. consider the origin in perturbation theory of a diagram XtY. This diagram is proportional (see fig. 1f) to one particular contraction of

$$\int \mathrm{d}5\, \mathrm{d}6\, \langle \mathrm{T}\{356^+1^+[\mathrm{H}_x]642^+5^+[\mathrm{H}_y]\}\rangle . \tag{6}$$

One can (anti)commute the 55^+ and 66^+ together inside the ordered product and rearrange to get the same contraction from

$$\pm\langle \mathrm{T}\{342^+1^+[\mathrm{H}_x][\mathrm{H}_y]\}\rangle . \tag{7}$$

Hence combining two Γ diagrams with the t operation and multiplying by ±1 produces another Γ diagram. Products of X and Y yielding a-type diagrams are given by

$$\begin{aligned}(\mathrm{X}\circ\mathrm{Y})_a = {} & \mathrm{X}_a\mathrm{s}\mathrm{Y}_a \pm \mathrm{X}_a\mathrm{s}'\mathrm{Y}_a \\ & \pm\mathrm{X}_a\mathrm{t}\mathrm{Y}_a \pm \mathrm{X}_b\mathrm{t}\mathrm{Y}_{b'} + \mathrm{X}_a\mathrm{t}'\mathrm{Y}_{b'} + \mathrm{X}_b\mathrm{t}'\mathrm{Y}_a \\ & \pm\mathrm{X}_a\mathrm{u}\mathrm{Y}_{c'} \pm \mathrm{X}_c\mathrm{u}\mathrm{Y}_a + \mathrm{X}_a\mathrm{u}'\mathrm{Y}_a + \mathrm{X}_c\mathrm{u}'\mathrm{Y}_{c'} .\end{aligned} \tag{8}$$

The other two types come by making the simultaneous cyclic permutations (s → t → u) and (a ← b ← c). A channel operation connecting two sums of diagrams is defined in terms of its actions on individual types.

Table 1
The action of exchange operations.

	s-exchange	t-exchange	u-exchange
Γ_a	$\pm\Gamma_a$	$\pm\Gamma_b$	$\pm\Gamma_c$
Γ_b	$\pm\Gamma_c$	$\pm\Gamma_a$	$\pm\Gamma_b$
Γ_c	$\pm\Gamma_b$	$\pm\Gamma_c$	$\pm\Gamma_a$

Consider now the crossing symmetric equations in the form of

$$S = \tfrac{1}{4}(I \pm I^s + T + U)(s \pm s')(I \pm I^s + S + T + U) \quad (9)$$

and its equivalent in other channels obtained by permuting ($S \to T \to U$) and ($s \to t \to u$) as in eq. (8). Starting with a given I, this will generate S, T, and U diagrams with similar structure. From the symmetry between the s, t, and u channels, a useful identity emerges. Consider, e.g. XuY. By rearranging the diagram and using s-exchange, we show, in fig. 1g, that

$$XuY = \pm(X't'Y')^s . \quad (10)$$

Generalizations of this result are obtained by simultaneously changing the primed–unprimed nature of the two channel operators or by permuting (s, t, u).

Hence any u-channel reducible diagram is the s-channel exchange of a t'-channel reducible diagram, within a sign. It is straightforward to show that in any order in perturbation,

$$U = \pm[T]^s = \pm[S]^t \quad (11)$$

and a host of similar relations. This simplifies the equation for S to

$$S = (I + T)s(I \pm I^s + S + T + U) ,$$
$$T = (I + S)t(I \pm I^t + S + T + U) . \quad (12)$$

In this form, the U can be obtained using eq. (11), leaving a set of two coupled equations for S and T involving s-channel and t-channel operations; nonetheless, the solution of these equations is fully crossing-symmetric. (In fact, one can replace T by $\pm[S]^u$ and U by $\pm[S]^t$ to have an equation which involves only S).

$$S = (I \pm [S]^u)s(I \pm I^s + S \pm [S]^t \pm [S]^u) . \quad (13)$$

Either (12) or (13) generates faithfully all generalized parquet diagrams. Eq. (12) is reminiscent of FHNC-like equations and the boson parquet equations. Note that the phases associated with all internal loops are generated properly by these equations. It is appropriate to note here that the above could equally be done with a larger set of irreducible diagrams, including the various arrangements of the diagram shown in fig. 1h. (In this sense, butter and parquet diagrams can be blended to achieve a proper consistency.) The best way to do this is to treat each irreducible diagram as a skeleton diagram and replace each bare interaction line in it with the full Γ. This is reminiscent of replacing elementary diagram sums by "basic" bridge diagrams using dressed correlations. (These should not be confused with the bridge diagrams of ref. [6].)

Using our sign convention, the energy (thermodynamic potential at finite T) can be written in terms of a G^λ computed with a potential λV as

$$E = E_0 + \frac{1}{2}\int_0^1 d\lambda \int d^3x \int d^3x' G_a^\lambda(x, x', x, x')V(x, x') . \quad (14)$$

We return now to considering only the bare interaction in I. The proper self-energy may then be computed from I_a, I_a^s, and those ladders in S_a whose top rung is a V by connecting points 4 and 2 by a Green's function and multiplying by ±1. This method generates all possible contributions exactly once with no counting problems, as is most easily seen from the Euler angle [6]. All reducible diagrams of the form S, T, or U can be decomposed into two boxes as shown in fig. 1i. After closing off one external line in any way, either two or three Green's functions connect the two boxes. If there are two, then the diagram could be obtained by using a *dressed* Green's function in the skeleton diagram obtained by solving (13). If three, then the lines can be arranged in two distinct ways. Both ways will be picked up by closing off the S_a and I_a diagrams. The proper self-energy obtained this way may then be used to calculate a modified Green's function to be used in the s, t, and u operations. This may be iterated to obtain consistency.

The energy may be computed directly from the self-energy, as outlined in ref. [1]. As in that case, the functional derivative of the energy diagrams generated in this process gives exactly

the self-energy. The proof is very similar to that given in ref. [1]; the diagrams have a more complicated structure, but the crossing symmetry allows the points associated with a vertex diagram to be redistributed at will.

On the other hand, the self-energy is not consistent with the vertex; e.g., the self-energy contribution obtained by closing the irreducible diagram in fig. 1h can be re-opened to give a reducible vertex diagram. However, the contribution of this reducible diagram to the self-energy is correctly generated by eqs. (12) or (13) and the self-energy prescription. There thus remains the problem of consistently calculating the self-energy when a larger class of irreducible diagrams is considered.

These diagram summations greatly extend the scope of the boson parquet equations. The two-particle vertex acquires a consistent set of exchange and butter diagrams, while the self-energy includes the exchange diagrams properly. Interestingly, the counting arguments used here are *simpler* than those used previously, a result of using the completely symmetric set of equations.

We have used these equations to generate all vertex and self-energy diagrams through fifth order. Through that order, we have verified by explicit calculation that all factors are correct and that each diagram appears exactly once. There are (60, 492, 5004, 56940) vertex diagrams and (2, 10, 82, 834) self-energy diagrams in order (2, 3, 4, 5), not counting self-energy insertions on internal lines.

There are many obvious applications for the present set of equations. We have previously shown how the boson parquet equations resemble optimized HNC [1]. We firmly believe that an FHNC-like theory is embedded in the present structure, and the generation of approximation schemes at the FHNC level is of great interest. The present approach generates a vertex with the symmetries of the full vertex for use in Landau theory. The study of equations with similar structure at finite temperature is an appealing prospect. Finally, the possibility of going beyond the approximations needed to reduce the full theory to the FHNC level is enticing.

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